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On the Uniqueness Property of Minimal Projectors*

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0. INTRODUCTION

Let G be a finite or infinite compact Abelian group with normed Haar measure m, let $C(G)$ be the space of all complex-valued continuous functions on G and let $\hat{G}$ be the set of all finite-dimensional, irreducible, unitary representations of G. So all elements of $\hat{G}$ are one-dimensional, multiplicative characters of G.

We consider finite $K \subset \hat{G}$, their linear hull $\langle K \rangle \subset C(G)$, and the Fourier projector $F_K: C(G) \to \langle K \rangle$;

$$(F_K f)(x) := \int_G D_K(x-t)\, f(t)\, dm(t)$$

with the Dirichlet kernel $D_K := \sum_{\gamma \in K} \gamma$.

The space of continuous linear operators $L(C(G), \langle K \rangle)$ is furnished with the norm

$$|\phi| := \sup_{|f|=1} |\phi f|$$

induced by the norm of uniform convergence on $C(G)$.

In the classical case $G = \mathbb{R}/2\pi\mathbb{Z}$, $K_n = \{e_k \mid e_k(t) = e^{ikt},\ |k| \leqslant n\}$, and $\langle K_n \rangle = P_n$, the problem of finding all minimal projectors is completely solved; Lozinsky [7] has shown that F_{K_n} has minimal norm among all linear projectors $C_{2\pi} \to P_n$, the uniqueness has been proved by Cheney *et al.* [1].

The last result does not extend to the general case according to Lambert [4, 5]. In the case $G = \mathbb{R}/2\pi\mathbb{Z}$, D_K real and determined up to a constant

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factor by its alternating points, F_K is the unique minimal norm projector, if and only if

$$\int_G |D_K(t)|\, \gamma(t)\, dt = 0 \qquad \text{for all } \gamma \in \hat{G} \backslash \{\gamma_1 - \gamma_2 \mid \gamma_1, \gamma_2 \in K\}.$$

In the general case this equivalence does not hold, and we give a characterization of the minimal operators (Section 2).

We begin with an investigation of the mean value

$$\int_G \lambda(x)\, dm(x)$$

with the Lebesgue function $\lambda(x) := |\hat{x}\phi|$; $\hat{x}$ denoting the evaluation functional $\hat{x}f := f(x)$. Doing this we prove a more general version of a result in [8]. The method of proof also is similar to the one used in [8].

1. On the Minimality in Mean

First we introduce some notations.

Of special importance in this paper is the convex cone

$$\Delta_K := \{ g \in \langle K \rangle \mid g(x) \cdot \overline{D_K(x)} \geqslant 0 \text{ for all } x \in G \}.$$

Let $M(G)$ be the space of all Radon measures on G. For each $\mu \in M(G)$ and $f, g \in C(G)$ we write

$$\mu(f) = \int_G f(t)\, d\mu(t)$$

and

$$(g\mu)(f) = \int_G f(t)\, g(t)\, d\mu(t).$$

Denoting a linear operator ϕ of the form

$$(\phi f)(x) = \int_G f(t)\, P(x-t)\, d\mu(t) \qquad \text{for all } f \in C(G)$$

by

$$\phi = \mu \otimes P,$$

all operators $\phi \in L(C(G), \langle K \rangle)$ have a representation of the form $\phi = \sum_{\gamma \in K} \mu_\gamma \otimes \gamma$.

We restrict our attention to operators of the form

$$\phi = \sum_{\gamma \in K} \mu_\gamma \otimes \gamma, \quad \mu_\gamma(G) = 1 \qquad \text{for all } \gamma \in K.$$

In this way all linear projectors are covered, as can be easily seen.

Let us define a norm preserving, linear extension ϕ^* of $\phi = \sum_{\gamma \in K} \mu_\gamma \otimes \gamma$ by

$$(\phi^* f)(x) := \sum_{\gamma \in K} \int_G f(t)\, \gamma(x-t)\, d\mu_\gamma(t)$$
$$\text{for all } f \in L^\infty(G),\ x \in G \tag{1.1}$$

Clearly, for each $x \in G$ we have

$$|\hat{x}\phi^*| = |\hat{x}\phi|. \tag{1.2}$$

For operators of the form $\phi = \sum_{\gamma \in K} \mu_\gamma \otimes \gamma$, $\mu_\gamma(G) = 1$ for all $\gamma \in K$ the Marcinkiewicz–Berman equality holds (compare [3]):

$$\int_G T_x \phi T_{-x} f\, dm(x) = F_K f \qquad \text{for all } f \in C(G) \tag{1.3}$$

with the translation operators

$$(T_x f)(t) = f(t+x).$$

By reasons of continuity (1.3) also holds for all $f \in L^\infty(G)$. The proof of (1.3) is nearly the same as in [3] and is omitted. Denoting $\sigma(t) := \operatorname{sgn} D_K(t)$ for all $t \in G$, by (1.3) we get

$$\int_G (T_x \phi^* T_{-x} \sigma)(0)\, dm(x) = (F_K^* \sigma)(0) = |F_K^*|, \tag{1.4}$$

and therefore

$$\int_G |\hat{x}\phi^*|\, dm(x) \geqslant |F_K^*|, \tag{1.5}$$

especially

$$|\phi| \geqslant |F_K|. \tag{1.6}$$

Now we give a characterization of those operators $\phi = \sum_{\gamma \in K} \mu_\gamma \otimes \gamma$, $\mu_\gamma(G) = 1$ for $\gamma \in K$, whose Lebesgue function is minimal in mean.

In this way we generalize our result [8] to Abelian groups and translation invariant subspaces.

THEOREM 1. *Let* $\phi = \sum_{\gamma\in K} \mu_\gamma \otimes \gamma$, $\mu_\gamma(G) = 1$ *for* $\gamma \in K$. *Then*

$$\int_G |\hat{x}\phi| \, dm(x) = |F_K| \tag{1.7}$$

holds, if and only if there is a positive Borel measure $\mu \in M(G)$, *and a family* $(K_t \in \Delta_K \mid t \in G)$, *the function* $(t, x) \mapsto K_t(x)$ *being two-dimensional Borel measurable and*

$$\phi f = \int_G f(t)\, T_{-t} K_t \, d\mu(t) \qquad \text{for all } f \in C(G). \tag{1.8}$$

Proof. Because $\langle K \rangle$ consists of continuous functions only, the Lebesgue function also is continuous and from (1.4) results (after extending ϕ to ϕ^*) Eq. (1.7) being equivalent to

$$(T_x \phi^* T_{-x} \sigma)(0) = |\hat{x}\phi^*| \qquad \text{for } m\text{-almost all } x \in G. \tag{1.9}$$

In the theorem, therefore, we can substitute (1.9) for (1.7). From (1.8) we get (again after extending ϕ to ϕ^*)

$$\begin{aligned}(T_x \phi^* T_{-x} \sigma)(0) &= \int_G \overline{\sigma(x-t)}\, K_t(x-t)\, d\mu(t) \\ &= \int_G |K_t(x-t)| \, d\mu(t) \\ &= |\hat{x}\phi^*|.\end{aligned}$$

Now we prove the reversal: If $\phi = \sum_{\gamma\in K} \mu_\gamma \otimes \gamma$ the Radon–Nikodym theorem yields a derivate d_γ of μ_γ for each $\gamma \in K$ with respect to $\mu := \sum_{\gamma\in K} |\mu_\gamma|$. Denoting $K_t := \sum_{\gamma\in K} d_\gamma(t) \cdot \gamma$ we have

$$\begin{aligned}(\phi f)(x) &= \sum_{\gamma\in K} \int_G f(t)\, \gamma(x-t)\, d\mu_\gamma(t) \\ &= \int_G f(t) \sum_{\gamma\in K} \gamma(x-t)\, d_\gamma(t)\, d\mu(t) \\ &= \int_G f(t) (T_{-t} K_t)(x)\, d\mu(t),\end{aligned}$$

and

$$(T_x\phi^* T_{-x}\sigma)(0) = \int_G \overline{\sigma(x-t)}\, K_t(x-t)\, d\mu(t).$$

On account of

$$|\hat{x}\phi^*| = \int_G |K_t(x-t)|\, d\mu(t)$$

from (1.9) we get

$$\int_G \overline{\sigma(x-t)}\, K_t(x-t)\, d\mu(t)$$
$$= \int_G |K_t(x-t)|\, d\mu(t) \qquad \text{for } m\text{-almost all } x \in G$$

respectively

$$\int_G \operatorname{Re}(\overline{\sigma(x-t)}\, K_t(x-t))\, d\mu(t)$$
$$= \int_G |K_t(x-t)|\, d\mu(t) \qquad \text{for } m\text{-almost all } x \in G. \tag{1.10}$$

Let

$$\mathscr{A} := \{(x,t) \in G^2 \mid \operatorname{Re}(\overline{\sigma(x-t)}\, K_t(x-t)) < |K_t(x-t)|\},$$
$$\mathscr{A}^t := \{x \in G \mid (x,t) \in \mathscr{A}\},$$
$$\mathscr{A}_x := \{t \in G \mid (x,t) \in \mathscr{A}\}.$$

Then by (1.10),

$$\mu(\mathscr{A}_x) = 0 \qquad \text{for } m\text{-almost all } x \in G. \tag{1.11}$$

The set $\mathscr{M} := \{t \in G \mid m(\mathscr{A}^t) > 0\}$ is Borel measurable. Let

$$\chi_{\mathscr{A}}(x,t) := 1 \qquad \text{if} \quad (x,t) \in \mathscr{A}$$
$$:= 0 \qquad \text{if} \quad (x,t) \notin \mathscr{A},$$

then by (1.11), $\int_G \mu(\mathscr{A}_x)\, dm(x) = 0$.

Fubini's theorem yields

$$\int_G \int_G \chi_{\mathscr{A}}(x, t)\, d\mu(t)\, dm(x) = \int_G \int_G \chi_{\mathscr{A}}(x, t)\, dm(x)\, d\mu(t)$$

respectively

$$0 = \int_G \mu(\mathscr{A}_x)\, dm(x) = \int_G m(\mathscr{A}^t)\, d\mu(t)$$

$$= \int_{\mathscr{M}} m(\mathscr{A}^t)\, d\mu(t),$$

and therefore

$$\mu(M) = 0. \tag{1.12}$$

For all $(t, x) \in \mathscr{M} \times G$, $K_t(x)$ can be replaced by 0 without modifying ϕ. Now we have

$$K_t(x) \cdot \overline{D_K(x)} \geqslant 0 \qquad \text{for } m\text{-almost all } x \in G, \tag{1.13}$$

because $m(\mathscr{A}^t) = 0$ for each $t \in G \backslash \mathscr{M}$.

Because of the continuity of $K_t \cdot D_K$, (1.13) holds for all $x \in G$, and therefore

$$K_t \in \Delta_K \qquad \text{for all } t \in G \backslash \mathscr{M}. \tag{1.14}$$

By reason of (1.12) condition (1.14) can be satisfied for all $t \in G$ without the measurability of the function $t \mapsto T_{-t} K_t$ being hurt.

This completes the proof of Theorem 1. ∎

2. Projectors of Minimal Norm

We want to determine those operators of the form $\phi = \sum_{\gamma \in K} \mu_\gamma \otimes \gamma$, $\mu_\gamma(G) = 1$ for $\gamma \in K$, which are of minimal norm, i.e., those ϕ's, for which

$$|\phi| = |F_K| \tag{2.1}$$

holds. Because by (1.5) $|F_K| \leqslant \int_G |\hat{x}\phi|\, dm(x) \leqslant |\phi|$, we have to consider only such operators ϕ which satisfy $\int_G |\hat{x}\phi|\, dm(x) = |F_K|$. By (1.9) this condition is equivalent to

$$(\phi^* T_{-x} \sigma)(x) = |\hat{x}\phi^*| \qquad \text{for } m\text{-almost all } x \in G, \tag{2.2}$$

and because of the continuity of the Lebesgue function, (2.1) yields

$$|\phi| = |F_K| \Leftrightarrow |\hat{x}\phi| = |F_K| \qquad \text{for all } x \in G. \tag{2.3}$$

Because of Theorem 1 each $\phi \in M$ has a representation of the form $(\phi f)(x) = \int_G f(t)\, K_t(x-t)\, d\mu(t)$ for all $f \in C(G)$ with a positive Borel measure μ and a family $(K_t \in \Delta_K \mid t \in G)$. So we get

$$|\hat{x}\phi| = \int_G |K_t(x-t)|\, d\mu(t),$$

and have proved the following

THEOREM 2. *Let $\phi = \sum_{\gamma \in K} \mu_\gamma \otimes \gamma$, $\mu_\gamma(G) = 1$ for $\gamma \in K$. Then*

$$|\phi| = |F_K|$$

holds if and only if ϕ has a representation of the form

$$(\phi f)(x) = \int_G f(t)\, K_t(x-t)\, d\mu(t) \qquad \textit{for all } f \in C(G)$$

with a positive Borel measure μ and a family $(K_t \in \Delta_K \mid t \in G)$ such that the function $(t, x) \mapsto K_t(x)$ is two-dimensional Borel measurable and $\int_G |K_t(x-t)|\, d\mu(t) = |F_K|$ holds for all $x \in G$.

3. THE CLASSICAL CASE $G = \mathbb{R}/2\pi\mathbb{Z}$

We investigate the case $G = \mathbb{R}/2\pi\mathbb{Z}$ especially with a $K \subset \hat{G}$ yielding a real Dirichlet kernel. To $\mathbb{R}/2\pi\mathbb{Z}$ belongs the set of characters $\hat{G} = \{e_j \mid j \in \mathbb{Z}, e_j(t) := e^{ijt}\}$.

Keeping in mind that the functions $\{\operatorname{Im} e_j = -\operatorname{Im}(e_{-j}) \mid j \in \mathbb{N} - \{0\}\}$ are linearly independent, we see that $D_K := \sum_{e_j \in K} e_j$ is real if and only if $K \subset \hat{G}$ is symmetrical, i.e., $(e_j \in K \Rightarrow e_{-j} \in K$ for all $j \in \mathbb{Z})$. Moreover, if K is symmetrical, D_K is even,

$$D_K(t) = D_K(-t) \qquad \text{for all } t \in G, \tag{3.1}$$

and especially the set of the zeros of D_K is symmetrical to 0. Now we look for functions $g \in \langle K \rangle$, such that

$$D_K + g \in \Delta_K. \tag{3.2}$$

These g's are also real functions, and therefore, have a representation of the form

$$g = \sum_{\substack{e_j \in K \\ j \geqslant 0}} (\alpha_j e_j + \bar{\alpha}_j e_{-j}) \qquad \text{with suitable } \alpha_j = x_j + iy_j, \quad x_j, y_j \in \mathbb{R},\ \alpha_0 = x_0,$$

$$= 2 \sum_{\substack{e_j \in K \\ j \geqslant 0}} \operatorname{Re}(\alpha_j e_j) = 2 \sum_{\substack{e_j \in K \\ j \geqslant 0}} x_j \operatorname{Re} e_j - y_j \operatorname{Im} e_j.$$

So, g is in the real space $U_0 = \langle \operatorname{Re} e_j, \operatorname{Im} e_j \mid e_j \in K \rangle$ with $\dim U_0 = \# K$. If D_K has p zeros counting multiplicities, and we look for functions $g \in U_0$, having the same zeros with at least the same multiplicities as D_K, we get a real subspace $U \subset \operatorname{lin} \Delta_K$ with $\dim U \geqslant \# K - p$. Let $A_1, A_2, ..., A_d$ be a basis of U with $A_1 = D_K$ and $A_i \in \Delta_K$ for $i \in \{2, ..., d\}$. Denoting $\varepsilon_i^* := \max\{\xi \mid \xi \cdot |A_i(t)| \leqslant (1/d)\, |D_K(t)| \text{ for all } t \in G\}$ we have $\varepsilon_i^* > 0$ for each $i \in \{1, ..., d\}$.

Now, we consider the following operators:

For $s \in \mathbb{N}$ and $\varepsilon_i \in \mathbb{R}$ with $|\varepsilon_i| < \varepsilon_i^*$ $(1 \leqslant i \leqslant d)$ let $P_s := F_K + \mu_s \otimes \sum_{i=1}^d \varepsilon_i A_i$ with $\mu_s := c_s m$ $(c_s(t) := \cos(st))$; then by Theorem 1 $P_s \in M$ holds.

Moreover, for all $x \in G$ we have

$$|\hat{x} P_s| = (\bar{P}_s T_{-x} \sigma)(x)$$

$$= |\hat{x} F_K| + \sum_{i=1}^d \varepsilon_i \int_G \overline{\sigma(x-t)}\, A_i(x-t) \cos(st)\, dm(t)$$

$$= |\hat{x} F_K| + \sum_{i=1}^d \sum_{j \in \mathbb{Z}} \varepsilon_i \alpha_{i,j} \int_G \overline{e_j(t)} \cos(st)\, dm(t)\, e_j(x),$$

with $\alpha_{i,j} := \int_G \overline{e_j(t)}\, |A_j(t)|\, dm(t)$,

$$|\hat{x} P_s| = |\hat{x} F_K| + \tfrac{1}{2} \left(\sum_{i=1}^d \varepsilon_i \alpha_{i,s} \right) e_s(x) + \tfrac{1}{2} \left(\sum_{i=1}^d \varepsilon_i \alpha_{i,-s} \right) e_{-s}(x);$$

altogether we get

$$|\hat{x} P_s| = |\hat{x} F_K| \qquad \text{for all } x \in G$$

$$\Leftrightarrow \left(\sum_{i=1}^d \varepsilon_i \alpha_{i,s} = 0 \text{ and } \sum_{i=1}^d \varepsilon_i \alpha_{i,-s} = 0 \right). \tag{3.3}$$

If $d \geqslant 3$ there are solutions $(\varepsilon_1, ..., \varepsilon_d) \neq (0, ..., 0)$ of the equations at the right side of (3.3), and the convex hull of these minimal operators also consists of minimal operators. We can find other minimal operators by choosing other μ_s's, e.g., $\mu_s = (\operatorname{Im} e_s) \cdot m$. In any case, if $d \geqslant 3$ we have found an infinite-dimensional, convex set of minimal operators.

For $s > 2 \cdot \max\{|j| \mid e_j \in K\}$ these P_s's even are projectors onto $\langle K \rangle$, because of

$$\left(\left(\mu_s \otimes \sum_{i=1}^{d} \varepsilon_i A_i\right) e_j\right)(x)$$
$$= \sum_{i=1}^{d} \int_G \varepsilon_i e_j(t) A_i(x-t) \cos(st)\, dm(t) = 0$$

for all $e_j \in K$. Therefore, the set of minimal projectors onto $\langle K \rangle$ also is infinite dimensional. But even for $d = 1$, more projectors than F_k of minimal norm may exist, as was shown already by Lambert [5, 6]. For if $d = 1$, then $A_1 = D_K$, and if there is a $s \in \hat{G} \setminus \{k_1 - k_2 \mid k_1, k_2 \in K\}$ with $\alpha_{1,s} = 0$, the equations in (3.3) are fulfilled also for $\varepsilon_1 \neq 0$.

As a simple example we consider a $K \subset \{e_j \in \hat{G} \mid j \in 2\mathbb{Z}\}$. Then $\operatorname{abs}(D_K)$ is π-periodical as D_K, and we get

$$\alpha_{1,s} = 0 \qquad \text{for odd } s.$$

4. Examples

In the classical case $\langle K \rangle = P_n$ being the space of all trigonometric polynomials of degree $\leqslant n$, we have $\dim \Delta_K = 1$, and the Fourier coefficients of $\operatorname{abs}(D_K)$ are all different from zero. This yields the uniqueness of the Fourier projector [1, 4, 8].

Apart from $K_n := \{e_j \mid |j| \leqslant n\}$, let us consider $K_n^* := \{e_j \in K_n \mid |j| \neq 1\}$, and compare these two cases.

We have $\dim\langle K_n \rangle = 2n + 1$ and $\dim\langle K_n^* \rangle = 2n - 1$. The classical Dirichlet kernel $D_n := D_{K_n}$ has a representation $D_n(t) = \sin((2n+1)t/2)/\sin(t/2)$, and for $D_n^*(t) := D_{K_n^*}(t)$ we get

$$D_n^*(t) = \sin((2n+1)t/2)/\sin(t/2) - 2\cos t. \tag{4.1}$$

In the following we get an upper bound of the number of zeros of D_n^* by comparing D_n^* with D_n. As both functions are real, with the aid of the considerations in Section 3, we can show that D_n^* does not have enough zeros for $n \in \mathbb{N} \setminus \{0, 1, 2, 3, 5\}$, and get an infinite-dimensional set of minimal projectors. We transform D_n and D_n^* into real polynomials of degree n by substituting $u = \cos t$, and denote $d_n(u) := D_n(t)$, and $d_n^*(u) := D_n^*(t)$. If $a < b$, and $d_n(a) \neq 0 \neq d_n(b)$, and $d_n^*(a) \neq 0 \neq d_n^*(b)$, we denote the number of zeros of $d_n [d_n^*]$ in (a, b) counting multiplicities by $N_n(a, b) [N_n^*(a, b)]$, and the number of sign changes in the sequences $(d_n(a), d_n'(a), ..., d_n^{(n)}(a))$ [respectively $(d_n^*(a), d_n^{*\prime}(a), ..., d_n^{*(n)}(a))$], the zeros being dropped, by

$W_n(a)|W_n^*(a)|$. Because d_n has only real zeros, for $a < b$ and $d_n(a) \neq 0 \neq d_n(b)$ we have

$$N_n(a, b) = W_n(a) - W_n(b), \tag{4.2}$$

and

$$n = N_n(-1, 1) = W_n(-1) - W_n(1), \tag{4.3}$$

as all zeros of d_n are in $(-1, 1)$. Thus, $W_n(-1) = n$, and $W_n(1) = 0$. By $d_n(x_k) = 0$ for $x_k := \cos(2\pi k/(2n+1))$, and $0 < k \leqslant n$, we get

$$N_n(\cos\theta, 1) = \lfloor (2n+1)\theta/(2\pi) \rfloor \qquad \text{for} \quad 0 < \theta \leqslant \pi, \text{ and } D_n(0) \neq 0.$$

Now, $N_n(\cos\theta, 1) = W_n(\cos\theta) - W_n(1) = W_n(\cos\theta)$ yields

$$W_n(\cos\theta) = \lfloor (2n+1)\theta/(2\pi) \rfloor. \tag{4.4}$$

Because of $|D_n(t)| \leqslant (\sin(t/2))^{-1}$ for all $t \in (0, \pi)$, and

$$(\sin(t/2))^{-1} < -2\cos t \qquad \text{for} \quad t \in [0.7\pi, \pi],$$

we have

$$d_n^*(x) \neq 0 \qquad \text{for} \quad x \in [-1, \cos(0.7\pi)].$$

For $-1 < x \leqslant \cos(0.7\pi)$, and $d_n^*(x) \neq 0$, therefore,

$$\begin{aligned} N_n^*(-1, 1) &= N_n^*(x, 1) \\ &\leqslant W_n^*(x) - W_n^*(1) \\ &\leqslant W_n^*(x) \end{aligned} \tag{4.5}$$

holds. Now, we will estimate $W_n^*(x)$ by comparison with $W_n(x)$. We have $d_n^*(x) = d_n(x) - 2x$, $d_n^{*\prime}(x) = d_n'(x) - 2$, and $d_n^{*(\nu)}(x) = d_n^{(\nu)}(x)$ for $2 \leqslant \nu \leqslant n$, so $W_n(x) - 2 \leqslant W_n^*(x) \leqslant W_n(x) + 2$ for any n. Because of (4.4) $W_n(\cos(0.7\pi)) = \lfloor 0.7n + 0.35 \rfloor$ holds, and for $n \geqslant 12$,

$$W_n^*(x) \leqslant W_n(x) + 2 \leqslant \lfloor 0.7n + 0.35 \rfloor + 2 \leqslant n - 2$$

yields by (4.5)

$$N_n^*(-1, 1) \leqslant n - 2.$$

So, for $n \geqslant 12$, D_n^* has at most $2n - 4$ zeros counting multiplicities, while $\# K_n^* = 2n - 1$, and by Section 3 we get

$$\dim(\operatorname{lin} \Delta_{K_n^*}) > 1.$$

More exactly we have $N_n^*(-1, 1) \leqslant 0.7n + 2.35$, or $\dim(\operatorname{lin} \Delta_{K_n^*}) \geqslant 0.6n - 5.7$. If $n \in \{4, 6, 7, 8, 9, 10, 11\}$, a more particular estimation of $W_n^*(x)$ is necessary to show that D_n^* has too small a number of zeros. In the simplest case $n = 4$ one can show that D_4^* has two distinct pairs of zeros only, and therefore

$$\dim(\operatorname{lin} \Delta_{K_4^*}) \geqslant 3.$$

If $s > 8$, the linear projectors

$$P_s := F_{K_4^*} + \sum_{i=1}^{3} \varepsilon_{s,i}(c_s m) \otimes A_i$$

with suitable $(\varepsilon_{s,1}, \varepsilon_{s,2} \varepsilon_{s,3})$ and $A_i \in \Delta_{K_4^*}$ are of minimal norm.

Note. When this work was finished, we were made aware of the recent work of S. D. Fisher, P. D. Morris, and D. E. Wulbert, "Unique Minimality of Fourier Projections." There it is shown that, in the case $G = \mathbb{R}/2\pi\mathbb{Z}$, the range of F_K having finite codimension and K being symmetrical, F_K is the unique minimal norm projection if and only if D_K is "determined by its zeros."

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